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COHERENCE FOR ASSOCIATIVITY NOT AN ISOMORPHISM

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§ 1. Introduction

Let C be a category, $\# : C \times C \to C$ a functor and

$$\alpha : 1_C \# (1_C \# 1_C) \to (1_C \# 1_C) \# 1_C$$

a natural transformation. A coherence theorem for the case in which α is an isomorphism is well known [6–8]. The aim of this note is to prove a coherence theorem for the case in which α is a natural transformation.

Our coherence theorem states the conditions in which every diagram of the form

$$\begin{tikzcd}
 & a_1 \arrow[r] & a_2 \arrow[r] & \dots \arrow[r] & a_n \arrow[dr] & \\
a \arrow[ur] \arrow[dr] & & & & & b \\
 & b_1 \arrow[r] & b_2 \arrow[r] & \dots \arrow[r] & b_m \arrow[ur] &
\end{tikzcd}$$

is commutative, where the vertices are formal shapes of the identity functor and the arrows are instantiations of α. An accurate description of this diagram is given by the graph of all the pairs of elements $\langle a_i, a_{i+1} \rangle$ such that there is an instantiation of α going from a_i to a_{i+1}; this graph, denoted by V, is defined in Definition 2 using a suitable description of the vertices provided by Theorem 1. The free category **P** generated by V is constructed and it is the category of the paths of arrows of V, and there exists a functor, $F : \mathbf{P} \to C$, in which every element $\langle a_i, a_{i+1} \rangle$ of V is mapped to the corresponding instantiation of α.

The core of the proof is a criterion (Theorem 3) to determine when there is a path from one vertex to another. The conditions for the coherence are the pentagonal condition of [7] and the naturality of α.

The vertices of the graph V (formal shapes of 1_C) are defined by the free algebra with one generator and one binary operation: this structure is used already in [3], and has been studied in [2] and [4]. Moreover it appears as one of the simplest examples of universal algebras.

The coherence theorem contained in this paper is a particular case of a theorem of Bénabou [1], whose proof is omitted: there could be some relations between that proof and the method we follow. We thank G.M. Kelly who pointed out this fact.

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§ 2

Definition 1. Let A be the category of algebras with one binary operation and $G : \mathsf{A} \to \mathbf{Set}$ the forgetful functor. As an immediate consequence of the Freyd adjoint functor theorem [5] it is well known that G has a left adjoint $F : \mathbf{Set} \to \mathsf{A}$.

Let $\{*\}$ be the set with one element, $*$. We shall call $A = F\{*\}$ the algebra of nonassociative numbers. It can be defined directly by saying that A is a set with one binary operation such that

(i) $* \in A$;

(ii) for every algebra $B \in \mathsf{A}$ and every element u of B there is one and only one element f of the homomorphism set $\mathsf{A}(A, B)$ such that $f(*) = u$.

The following relations for elements of A are well known:

(i) $a + b = a' + b' \Rightarrow a = a' \wedge b = b'$,

(ii) $a \neq * \Rightarrow a = a'' + a'$.

We define the *norm* in A as the map from A onto N, the set of natural numbers, given by the conditions: $|*| = 1$, $|a + b| = |a| + |b|$.

Intuitively speaking, an element of A is an arrangement of parentheses located among the elements of a finite sequence of repetitions of $*$; we can assign to every occurrence of $*$ in a the number of parentheses in whose second component this occurrence appears and call this number the index of the occurrence. Thus the index is also the "number of second components of an addition" in which the occurrence appears. This suggests the representation of the element a as the ordered sequence of the indexes of the occurrences of $*$ and gives the intuitive justification of the following definition of w.

Let w be the function $w : A \to \bigcup_{m=1}^{\infty} N^m$ defined by the conditions

(i) $w(*) = (0)$.

(ii) $w(a) = (a_1, a_2, \ldots, a_n)$, $w(b) = (b_1, b_2, \ldots, b_m)$, imply
$w(a + b) = (a_1, \ldots, a_n, 1 + b_1, 1 + b_2, \ldots, 1 + b_m)$.

We will use the convention $\mathring{x} = -1 + x$.

Theorem 1. *The function w is injective and its image is the set of the sequences of integers $(a_1, a_2, \ldots, a_n)$ satisfying the following conditions:*

(i) $a_1 = 0$, *and* $a_i > 0$ *for* $i \neq 1$,

(ii) $a_{i+1} > a_i \Rightarrow a_{i+1} = 1 + a_i$.

Proof. It is easy to prove by induction on $|a|$ that $w(a)$ satisfies conditions (i) and (ii).

We are going to prove by induction on n that if $(a_1, a_2, \ldots, a_n)$ satisfies conditions (i) and (ii), then there is one and only one element a such that $w(a) = (a_1, a_2, \ldots, a_n)$: let r be the largest integer such that $a_r = 1$; then we have that if

$$w(a') = (a_1, a_2, \ldots, a_{r-1}),$$
$$w(a'') = (\mathring{a}_r, \mathring{a}_{r+1}, \ldots, \mathring{a}_n),$$

then

$$w(a' + a'') = (a_1, a_2, \ldots, a_n).$$

It is immediate that if $w(b' + b'') = w(a' + a'')$, then $wb' = wa'$ and $wb'' = wa''$: these relations and the induction hypothesis prove that $a = a' + a''$ is unique.

Remark 1. (a). In general, we shall identify A with the image of w.

(b). We shall identify $(a_1, a_2, \ldots, a_n)$ with the infinite sequence $(a_1, a_2, \ldots, a_n, 0, 0, \ldots)$. This will simplify the expression of some relations.

(c). When no doubt can arise we shall use the symbol $(x_1, x_2, \ldots)$ for the element wx or x

(d). If $a \neq *$ and $wa = (a_i)$, then $|a| = n$ is the largest i such that $a_i \neq 0$, as can be immediately proved.

Definition 2. Let p and q be natural numbers and $p \leq q$. We shall define the set $V_{p,q}$ as the subset of $A \times A$ consisting of all the elements $\langle (a_i), (b_i) \rangle$ such that

(i) $a_p < a_{p+1}, a_{p+2}, \ldots, a_q$;

(ii) $2 \leq a_p$;

(iii) $a_p > a_{q+1}$;

(iv) $a_i = b_i$ for $1 \leq i \leq p-1$ and $q + 1 \leq i$,
$\mathring{a}_i = b_i$ for $p \leq i \leq q$.

We now define V by

$$V = \bigcup_{\substack{p,q \in N \\ p \leq q}} V_{p,q}.$$

Remark that if $\langle a, b \rangle \in V_{p,q}$, then

$$p = \inf\{i \mid a_{i-1} = b_{i-1}\},$$
$$q = \sup\{i \mid a_i \neq b_i\},$$

and this proves that the sets $V_{p,q}$ are disjoint. It is clear from the definition that for every $x \in A$,

$$\langle a, b \rangle \in V \Rightarrow \langle a + x, b + x \rangle \in V \wedge \langle x + a, x + b \rangle \in V.$$

We shall indicate by P the free category generated by V considered as a graph: its arrows are sequences of the form

$$\{a = g_r, g_{r-1}, \ldots, g_2, g_1 = b\},\ r \geq 1,$$

where if $r \neq 1$, $\langle g_{i+1}, g_i \rangle \in V$ for every i: usually these sequences are represented in the form

$$a = g_r \to g_{r-1} \to \ldots \to g_1 = b.$$

Theorem 2. *Let $\langle a, b \rangle$ be an element of V; then one and only one of the following relations holds:*

(i). *For some $x \in A$, $\langle a, b \rangle = \langle x + a', x + b' \rangle$ and $\langle a', b' \rangle \in V$,*

(ii). *For some $x \in A$, $\langle a, b \rangle = \langle a' + x, b' + x \rangle$ and $\langle a', b' \rangle \in V$,*

(iii). *For some $x, y, z \in A$, $a = x + (y + z)$, $b = (x + y) + z$.*

Proof. Let $f_{p,q}$ be the function defined by $V_{p,q}$; that is,

$$f_{p,q}(x) = y \Leftrightarrow \langle x, y \rangle \in V_{p,q}.$$

Suppose that $\langle a, b \rangle \in V_{p,q}$, $a = a' + a''$, $|a'| = r$. Then $a_{r+1} = 1$ and $2 \leq a_p \leq a_{p+1}, \ldots, a_q$ imply that $r + 1 < p \leq q$ or $p \leq q < r + 1$.

If $r + 1 < p \leq q$ and $a_p > 2$, then $b = f_{p,q}(a' + a'') = a' + f_{p-r,q-r}(a'')$ and we are in case (i) of the theorem.

If $p \leq q < r$, then $b = f_{p,q}(a' + a'') = f_{p,q}(a') + a''$ and we are in case (ii) of the theorem.

If $r + 1 < p \leq q$ and $a_p = 2$, then $2 = a_p > a_{q+1}$. Note that $r = |a'|$ and $a = a' + a''$ imply that for $i > r + 1$, $a_i = 0$ or $a_i > 1$; hence, $a_{q+1} = 0$ and for $a'' = y + z$, $a = a' + (y + z)$, $b = (a' + y) + z$, and we are in case (iii) of the theorem.

From $\langle a, b\rangle \in V \Rightarrow a \neq b$, it follows easily that any two of the three relations of the theorem are incompatible.

Remarks 2. (a). Theorem 2 shows that V can also be defined as the smallest subset of $A \times A$ satisfying the following conditions:

(i) $\langle x + (y + z), (x + y) + z\rangle \in V$;

(ii) $\langle a, b\rangle \in V \Rightarrow \langle a + x, b + x\rangle \in V \wedge \langle x + a, x + b\rangle \in V$.

(b). Intuitively speaking, Theorem 2 states that the arrows in **P** are the paths obtained by "formal application of the associative law".

(c). The operation of A induces an operation on $A \times A$ given by $\langle u, v\rangle + \langle u', v'\rangle = \langle u + u', v + v'\rangle$. The above theorem states that any element is obtained, in a unique way, by the instantiation (that is, the composition in the operation of $A \times A$) of one element of the form $\langle x + (y + z), (x + y) + z\rangle$ with others of the form $\langle a_i, a_i\rangle$. In this sense the results of the theorem will be used in Definition 4.

Definition 3. Let M be the set of all pairs $\langle (a_i), (b_i)\rangle$ such that,

(i) $|(a_i)| = |(b_i)|$;

(ii) if for some $s < t$ we have $a_s < a_i$ for $s < i < t$ and $a_s \geqslant a_t$, then $b_s \geqslant b_t$.

Suppose that $\langle (a_i), (b_i)\rangle \in M$; we are going to indicate some relations between (a_i) and (b_i):

Remark 3. If for some $u < v$ we have $a_u \leqslant a_i$ for $u < i < v$ and $a_u \geqslant a_v$, then $b_u \geqslant b_v$.

Proof: Define a sequence $u = u_1 < u_2 \ldots < u_r = v$ such that it is possible to apply the definition of M for $s = u_i$, $t = u_{i+1}$.

Remark 4. If some $u < v$ we have $a_u \leqslant a_i$ for $u \leqslant i \leqslant v$, then $a_u - b_u \leqslant a_i - b_i$ for $u \leqslant i \leqslant v$.

This can be proved by induction on $v - u$. If $v - u > 1$ and for some j so that $u < j < v$ we have that $a_u = a_j$, then, by the induction hypothesis,

$$a_u - b_u \leqslant a_i - b_i \text{ for } u \leqslant i \leqslant j,$$

$$a_j - b_j \leqslant a_i - b_i \text{ for } j \leqslant i \leqslant v.$$

If $a_u < a_i$ for $u < i \leqslant v$, then $a_u < a_{u+1}$ and

$$a_u - b_u = a_{u+1} - 1 - b_u \leqslant a_{u+1} - b_{u+1},$$

and the induction hypothesis can immediately be applied. The only remaining case is $v - u = 1, a_u = a_v$ and this is a consequence of the definition of M applied for $s = u,\ t = u + 1$.

Remark 5. $a_i \geqslant b_i$ for every i. Take $u = 1$, $v = |(a_i)|$ and apply Remark 4.

An intuitive interpretation of the conditions of the definition of M is the following: an element a of A is a finite sequence of symbols $*$ with parentheses among them. Take a number v and choose the largest of all the parentheses occurrring in a that end exactly after the $*$ of the v^{th} place: we call $N_a(v)$ the number of symbols $*$ contained in the parentheses. Then $\langle a, b\rangle \in M$ if and only if for any v, $N_a(v) \leqslant N_b(v)$. In other words, define that a parenthesis P is "larger" than another P' when P and P' end on the right in the same place and $P \supset P'$; then $\langle a, b\rangle \in M$ if and only if for any parenthesis of a there exists at least one parenthesis of b that is "larger" than it in the above sense. We are not going to prove in detail the above statement which will not be used later. We want to note only that the proof is an easy consequence of the following facts:

Fact 1. If $a = (a_i)$, then the existence of a parenthesis in the expression of a from the $*$ in the u^{th} place to the one in the v^{th} place (including both elements) is equivalent to the relation $a_u \geqslant -1 + a_{v+1}$ and $a_u < a_i$ for $u < i \leqslant v$.

Fact 2. $\langle (a_i), (b_i)\rangle \in M$ if and only if the following relation holds for any pair $\langle u, v\rangle$:

$$a_i \geqslant a_v \text{ for } u \leqslant i \leqslant v \Rightarrow b_i \geqslant b_u \text{ for } u \leqslant i \leqslant v.$$

The proof of Fact 1 can be done by induction on the rank of a. The proof of Fact 2 uses a method similar to the one used in the proof of Remarks 3 and 4.

Theorem 3. *$\langle a, b\rangle \in M$ if and only if $\mathbf{P}(a, b) \neq \emptyset$.*

Proof. It is immediate that for any $x, y, z \in A$,

$$\langle x + (y + z), (x + y) + z\rangle \in M,$$

and

$$c \in A \wedge \langle a, b\rangle \in M \Rightarrow \langle a + c, b + c\rangle \in M \wedge \langle c + a, c + b\rangle \in M.$$

From this it follows that $V \subset M$.

To prove that $\mathbf{P}(a, b) \neq \emptyset \Rightarrow \langle a, b\rangle \in M$ it will be sufficient (within a trivial induction on the number of steps from a to b) to prove

$$\langle a, b\rangle \in V \wedge \langle b, c\rangle \in M \Rightarrow \langle a, c\rangle \in M.$$

Suppose for this that for $u < i < v$, $a_u < a_i$ and $a_u \geqslant a_v$. It is immediate that $b_u \leqslant b_i$ for $u < i < v$ and $b_u \geqslant b_v$ and from Remark 3 follows $c_u \geqslant c_v$.

Suppose that $\langle a, b\rangle \in M$, $a = (a_i)$, $b = (b_i)$, $a \neq b$. We are going to prove the existence of $a' \in A$ such that $\langle a, a'\rangle \in V$ and $\langle a', b\rangle \in M$ and, within an immediate induction on $\Sigma(b_i - a_i)$, which is positive by Remark 5, we can conclude that

$$\langle a, b\rangle \in M \Rightarrow \mathbf{P}(a, b) \neq \emptyset.$$

Define q by $q = \sup\{i \mid a_i > b_i\}$. Suppose that $a_q < a_{q+1}$, then

$$b_q \geqslant -1 + b_{q+1} = -1 + a_{q+1} = a_q > b_q,$$

a contradiction. If $a_q = a_{q+1}$, then $\langle a, b\rangle \in M$ implies $b_q \geqslant b_{q+1}$ and

$$b_{q+1} \leqslant b_q < a_q = a_{q+1} = b_{q+1},$$

a contradiction. Hence $a_q > a_{q+1}$ and the set U of all u such that

(i) $u \leqslant q$, $a_u - b_u = a_q - b_q$,

(ii) $a_u < a_i$ for $u < i \leqslant q$ and $a_u > a_{q+1}$,

is not empty because $q \in U$. Define $p = \inf(U)$, then

$$p \neq 1, b_p \geqslant b_{q+1} = a_{q+1}, \quad a_p > b_p \geqslant a_{q+1}, a_p \geqslant 2,$$

and we can define

$$a' = (a_i') = (a_1, \ldots, a_{p-1}, \mathring{a}_p, \ldots, \mathring{a}_q, a_{q+1}, \ldots).$$

Suppose that for $u < v$ we have the relations $a_u' < a_i'$ for $u < i < v$ and $a_u' \geqslant a_v'$ and it will be sufficient to prove that $b_u \geqslant b_v$.

If $p \leqslant q < u$, then for $u < i < v$

$$a_u = a_u' < a_i' = a_i, \; a_u = a_u' \geqslant a_v' = a_v,$$

and

$$\langle a, b\rangle \in M \Rightarrow b_u \geqslant b_v.$$

If $p \leqslant u \leqslant q$, then

$$a_u = a'_u + 1 \geqslant a'_v + 1 \geqslant a_v,$$

and the hypothesis $\langle a, b\rangle \in M$ allows us to reduce to the case in which for some i, $u < i < v$, $a_u \geqslant a_i$; in this case, $a_i = a'_i$ and $p \leqslant u \leqslant q < -1 + v$, hence

$$a_{q+1} < a_p \leqslant \mathring{a}_u + 1 = a'_u + 1 \leqslant a'_{q+1} = a_{q+1},$$

a contradiction.

If $u < p$, we have

$$a_u = a'_u < a'_i \leqslant a_i,$$

hence we can suppose that $a_u < a_v$ and then

$$a_u = a'_v = -1 + a_v, u < p \leqslant v \leqslant q.$$

If $u < p < v \leqslant q$, we have

$$a'_v \leqslant a'_u < a'_p < \mathring{a}_v = a'_v,$$

which is a contradiction. Hence, we are reduced to the case,

$$u < p = v \leqslant q, a_u = -1 + a_v = -1 + a_p \geqslant a_{q+1}$$

and

$$a_u \leqslant a_i \text{ for } u < i \leqslant q.$$

If $a_u = a_{q+1}$, then

$$a_p = 1 + a_u = 1 + a_{q+1}$$

and by condition (ii) of Definition 3,

$$b_u \geqslant b_{q+1} = a_{q+1} = -1 + a_q \geqslant b_q = b_v.$$

If $a_u > a_{q+1}$, using Remark 4 we have that

$$a_u - b_u \leqslant a_i - b_i \text{ for } u \leqslant i \leqslant q,$$

that is, $a_u - b_u \leqslant a_p - b_p$; but $a_u - b_u = a_p - b_p$ is a contradiction with the definition of p, hence $a_u - b_u < a_p - b_p$, and then

$$b_u > a_u - a_p + b_p = -1 + b_p = -1 + b_v,$$

and therefore $b_u \geqslant b_v$.

Remark 6. Theorem 3 is the statement of the following criterion:

The existence of a path from (a_i) to (b_i) is equivalent to the following conditions:

(i) $|(a_i)| = |(b_i)|$,

(ii) $a_s < a_{s+1}, a_{s+2}, \dots, a_{t-1} \wedge a_s \geqslant a_t \wedge s < t \Rightarrow b_s \geqslant b_t$.

Remark 7. In the proof of Theorem 3 there is included an algorithm to compute a path between two given elements.

Definition 4. Let C be a category, $\# : C \times C \to C$ a functor and

$$\alpha : 1_C \# (1_C \# 1_C) \to (1_C \# 1_C) \# 1_C$$

a natural transformation.

Remark that the functor $\#$ defines a functor, also indicated by $\#$, from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$, where $\mathcal{C}$ is the category whose objects are all the functors from C^n to C for $n = 1, 2, \dots,$ and whose arrows are the natural transformations among these functors.

Then, if

$$G : C^m \to C,$$
$$H : C^p \to C,$$
$$K : C^q \to C,$$

and

$$\alpha_{G,H,K} : G \# (H \# K) \to (G \# H) \# K$$

is defined by

$$\alpha_{G,H,K}(a_i) = \alpha_{G(a_i), H(a_{m+j}), K(a_{m+p+k})}, \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant p,\ 1 \leqslant k \leqslant q.$$

We are going to define a map $F : A \to \text{Ob } \mathcal{C}$, by the conditions

(i) $F(*) = 1_C$,

(ii) $F(a + b) = Fa \# Fb$.

The functor $\#$ defines an algebraic structure on Ob C and the universal property of A shows that F is uniquely defined.

It is immediate to prove, by induction on n, the existence for every n of a map

$$F_n : V \cap (A_n \times A_n) \to \mathrm{Arr}\ C,$$

where A_n is the set of all the elements of A with norm not greater than n such that

(i) $F_n \langle a + x, b + x \rangle = F_n \langle a, b \rangle \# 1_{Fx}$,

(ii) $F_n \langle x + a, x + b \rangle = 1_{Fx} \# F_n \langle a, b \rangle$,

(iii) $F_n \langle x + (y + z), (x + y) + z \rangle = \alpha_{Fx, Fy, Fz}$.

It is easy to show by induction on n not only the existence of F_n but also its uniqueness, and as a consequence of the uniqueness we have that

$$F_{m+n} \mid V \cap (A_n \times A_n) = F_n,$$

and

$$F = \mathsf{U}_n F_n$$

is a map from V to Arr C, that ordinarily we shall denote also by F, with the properties

(i) $F \langle a + x, b + x \rangle = F \langle a, b \rangle \# 1_{Fx}$,

(ii) $F \langle x + a, x + b \rangle = 1_{Fx} \# F \langle a, b \rangle$,

(iii) $F \langle x + (y + z), (x + y) + z \rangle = \alpha_{Fx, Fy, Fz}$.

There is no difficulty in proving that the two above maps, both denoted by F, define a morphism of graphs from V to C; this can be done by induction on n proving that $\langle F \mid A_n, F_n \rangle$ is a morphism of graphs from $V \cap (A_n \times A_n)$ to C, which we shall indicate by F, and which can be enlarged to a functor $F : \mathbf{P} \to C$.

We shall suppose also that in C the pentagonal condition holds, that is, that for any objects a, b, c and $d \in C$ the following diagram is commutative

$$\begin{tikzcd}
a \# [b \# (c \# d)] \arrow[r, "\alpha_{a,b,c \# d}"] \arrow[dd, "1_a \# \alpha_{b,c,d}"'] & (a \# b) \# (c \# d) \arrow[d, "\alpha_{a \# b, c, d}"] \\
 & [(a \# b) \# c] \# d \\
a \# [(b \# c) \# d] \arrow[r, "\alpha_{a, b \# c, d}"] & [a \# (b \# c)] \# d \arrow[u, "\alpha_{a, b, c} \# 1_d"']
\end{tikzcd}$$

This commutativity allows us to construct similar commutative diagrams in C.

Theorem 4. *Suppose that for* $a = (a_1, a_2, \dots, a_n)$, $b = (b_1, b_2, \dots, b_n)$, $c = (c_1, c_2, \dots, c_n)$ *we have* $a \to b$ *and* $a \to c$. *Then there exists an element* $d = (d_1, d_2, \dots, d_n)$ *and a diagram*

$$\begin{tikzcd} & b \arrow[dr, dashed] & \\ a \arrow[ur, dotted] \arrow[dr, dotted] & & d \\ & c \arrow[ur, dashed] & \end{tikzcd}$$

where the value by F *of the two paths from* a *to* d *is the same. Moreover, if for some pair* $s < t$, $d_s \geqslant d_t$ *and* $s < i < t \Rightarrow d_s < d_i$, *then at least one of the following relations holds:*

(i) $b_s \geqslant b_t \wedge s < i < t \Rightarrow b_s < b_i$;
(ii) $c_s \geqslant c_t \wedge s < i < t \Rightarrow c_s < c_i$.

Proof. We are going to prove the theorem for the special case

$$\begin{aligned} a &= x + [y + (z + w)], \\ b &= (x + y) + (z + w), \\ c &= x + [(y + z) + w]. \end{aligned}$$

The diagram required by the theorem is the following:

$$\begin{tikzcd}[column sep=small] & (x+y)+(z+w) \arrow[rr, dashed] & & {[(x+y)+z]+w} \\ x + [y + (z+w)] \arrow[ur] \arrow[dr] & & & \\ & x + [(y+z)+w] \arrow[r, dashed] & {[x+(y+z)]+w} \arrow[uur, dashed] & \end{tikzcd}$$

Let

$$x = (x_1, x_2, \dots, x_p),\ y = (y_1, y_2, \dots, y_q),$$

$$z = (z_1, z_2, \dots, z_r),\ w = (w_1, w_2, \dots, w_s),$$

then

$$\begin{aligned} b &= (x_i, 1 + y_i, 1 + z_i, 2 + w_i), \\ c &= (x_i, 1 + y_i, 2 + z_i, 2 + w_i), \\ d &= (x_i, 1 + y_i, 1 + z_i, 1 + w_i). \end{aligned}$$

Suppose now that for some $s < t$, $d_s \geqslant d_t$ and $s < i < t$, imply $d_s < d_i$; hence, if $s < t \leqslant p + q + r$, condition (i) holds. If $p + q < s < t$, condition (ii) holds. If

$$s < p + q + 1 < p + q + r + 1 \leqslant t,$$

then

$$d_s < d_{p+q+1} = 1 + z_1 = 1,$$

hence

$$d_s = d_t = 0, s = 1, t > p + q + r + s$$

(see Remark 1b), and (i) and (ii) hold.

We shall omit in the rest of the proof all the arguments that are similar to the above. Specifically, we can prove the cases indicated by the following diagrams:

$$\begin{tikzcd}
 & b = (x + y) + z \arrow[dr, dashed] & \\
a = x + (y + z) \arrow[ur] \arrow[dr] & & d = (x' + y') + z' \\
 & c = x' + (y' + z') \arrow[ur, dashed] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & b = x' + y \arrow[dr, dashed] & \\
a = x + y \arrow[ur, dashed] \arrow[dr] & & d = x' + y' \\
 & c = x + y' \arrow[ur, dashed] &
\end{tikzcd}$$

To prove the general case, we can use induction on $|a|$. If $a = x + y$, $b = x + y'$, $c = x + y''$, determine $\bar{y}$ using the induction hypothesis in

$$\begin{tikzcd}
 & y' \arrow[dr, dashed] & \\
y \arrow[ur] \arrow[dr] & & \bar{y} \\
 & y'' \arrow[ur, dashed] &
\end{tikzcd}$$

and take $d = x + \bar{y}$. If $a = x + y$, $b = x' + y$, $c = x'' + y$, take $d = \bar{x} + y$, with the $\bar{x}$ determined in a similar way. This and the above cases complete the induction proof.

Remark 8. In the conditions of Theorem 4, if for some $a' \in A$ we know that $\mathbf{P}(b, a') \neq \emptyset$ and $\mathbf{P}(c, a') \neq \emptyset$, then Theorem 3 insures that $\mathbf{P}(d, a') \neq \emptyset$.

Theorem 5 (Coherence theorem). *In the functor $F : P \to C$, the image of every arrow depends only upon the domain and codomain of the arrow.*

Proof. Let T and S be two paths from a to b

$$\begin{tikzcd}
 & t_1 \arrow[r] & t_2 \arrow[r] & \cdots \arrow[r] & t_{n-1} \arrow[r] & t_n \arrow[dr] & \\
a \arrow[ur] \arrow[dr] & & & & & & b \\
 & s_1 \arrow[r] & s_2 \arrow[r] & \cdots \arrow[r] & s_{m-1} \arrow[r] & s_m \arrow[ur] &
\end{tikzcd}$$

We are going to prove by induction on $\Sigma(a_i - b_i)$ that $F(S) = F(T)$. For this, it will be sufficient to prove the existence of an element a' and of the paths of the dotted arrows in the diagram

$$\begin{tikzcd}
 & t_1 \arrow[r] \arrow[dr, dashed] & t_2 \arrow[r] & \cdots \arrow[r] & t_{n-1} \arrow[r] & t_n \arrow[dr] & \\
a \arrow[ur] \arrow[dr] & & a' \arrow[rrrr, dashed] & & & & b \\
 & s_1 \arrow[r] \arrow[ur, dashed] & s_2 \arrow[r] & \cdots \arrow[r] & s_{m-1} \arrow[r] & s_m \arrow[ur] &
\end{tikzcd}$$

in such a way that the image by F of the two paths of the diagram in the left is the same, and this is a consequence of Theorems 4 and 3 (see Remark 8).

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